Painlevé Properties and Exact Solutions of the Generalized Coupled KdV Equations

Li-Jun Ye and Ji Lin

Department of Physics, Zhejiang Normal University, Jinhua 321004, China

Reprint requests to J. L.; E-mail: linji@zjnu.cn

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The generalized coupled Korteweg-de Vries (GCKdV) equations as one case of the four-reduction of the Kadomtsev-Petviashvili (KP) hierarchy are studied in details. The Painlevé properties of the model are proved by using the standard Weiss-Tabor-Carnevale (WTC) method, invariant, and perturbative Painlevé approaches. The meaning of the negative index $k = -2$ is shown, which is indistinguishable from the index $k = -1$. Using the standard and nonstandard Painlevé truncation methods and the Jacobi elliptic function expansion approach, some types of new exact solutions are obtained.

Key words: Painlevé Analysis; GCKdV Equations; Exact Solutions.

1. Introduction

Physical systems are frequently characterized by nonlinear differential equations. The integrability of a nonlinear partial differential equation (PDE) is an interesting topic in nonlinear science. The fact that the Painlevé analysis arises as a solving method of many nonlinear systems is known for quite some time [1]. As is known, the Painlevé analysis developed by Weiss-Tabor-Carnevale (WTC) [2] not only is one of the most powerful methods to prove the integrability of a model, but also can be used to find some exact solutions [3]. Later, the WTC approach has been generalized by R. Conte [4], A. Pickering [5], and S. Y. Lou [6] in some ways in order to find more exact and explicit solutions of nonlinear PDEs.

A partial differential equation is said to possess the Painlevé property, if the solutions of the PDE are single-valued about the movable singularity manifold. To be precise, if the singularity manifold is determined by

$$f(z_1,\ \dots,\ z_n) = 0, \tag{1}$$

and $u = u(z_1,\ \dots,\ z_n)$ is a solution of the PDE, then we assume that

$$u = f^{\alpha} \sum_{k=0}^{\infty} u_k f^k, \tag{2}$$

where $f = f(z_1,\ \dots,\ z_n)$ and $u_k = u_k(z_1,\ \dots,\ z_n)$ $(u_0 \neq 0)$ are analytic functions of the variables z_k in a neighborhood of the manifold (1), and α is an integer. Substituting (2) into the PDE determines the value of α and defines recursion relations for u_k $(k = 0,\ 1,\ 2,\ \dots)$. If the ansatz (2) is correct, the PDE is said to possess the Painlevé property and is eventually conjectured to be integrable.

Such an analysis first requires a choice of the expansion family (or branch). This implies a selection of the leading order exponent α and the leading order coefficient u_0. For each family, there is a set of indices, or resonances, which give the values of k for which arbitrary coefficients should be introduced into (2). Here a maximal family is used to denote any family with a number of indices equal to the order of the equation(s) being considered, while a principal family is any family with all the resonances of nonnegative integers, except -1, which should occur once only. As is known, the so-called standard Painlevé analysis [2] is suitable only for a PDE which has a maximal principle expansion family.

In case that there are not enough arbitrary coefficients, the Painlevé expansion (2) only represents a particular or even singular solution. This may happen for a number of reasons, a nonmaximal family, a noninteger index, or a negative integer index distinct from -1. In order to deal with such questions, the method presented in this paper is a perturbative Painlevé analysis [7], which guarantees an arbitrary coefficient for each index.

In this paper, we are devoted to study the Painlevé property for the generalized coupled Korteweg-de

Vries (GCKdV) equations. It is organized as follows. In Section 2, we explore the GCKdV equations by the standard Painlevé analysis, and find that the considered system has a nonprincipal but maximal expansion family. In Section 3, we give the corresponding arbitrary function for the negative integer $k=-2$ by using the perturbative Painlevé method, i. e., the remaining problem of the preceding section is settled. By applying some different expansion methods, several exact solutions for the GCKdV equations are given in Section 4. And the last section contains a short summary.

2. Standard and Invariant Painlevé Analysis for GCKdV Equations

It is shown that the GCKdV system,

$$\begin{aligned} &u_t - \frac{1}{4}u_{xxx} - 3uu_x - 3w_x + 6vv_x = 0, \\ &v_t + \frac{1}{2}v_{xxx} + 3uv_x = 0, \\ &w_t + \frac{1}{2}w_{xxx} + 3uw_x = 0, \end{aligned} \tag{3}$$

introduced by J. Satsuma and R. Hirota [8], is a special case of the four-reduced Kadomtsev-Petviashvili (KP) hierarchy. Using a bilinear transformation method, it has been studied by many authors [8, 9], who also have shown that the soliton solutions can be derived from those of the KP equation. According to the standard WTC method, if the system is Painlevé integrable, then all the possible solutions of the system can be represented as

$$u = \sum_{k=0}^{\infty} u_k f^{k+\alpha},\ v = \sum_{k=0}^{\infty} v_k f^{k+\beta},\ w = \sum_{k=0}^{\infty} w_k f^{k+\gamma}, \tag{4}$$

with sufficiently many arbitrary functions u_k, v_k, and w_k in addition to f, where $f = f(x,t)$, $u_k = u_k(x,t)$, $v_k = v_k(x,t)$, and $w_k = w_k(x,t)$ are analytical functions in the neighborhood of $f(x,\ t) = 0$, and α, β, and γ should be the negative integers. In other words, the solutions of the GCKdV equations are single-valued on an arbitrary singularity manifold f.

Substituting (4) into (3) and using a leading-order analysis, it is found that the GCKdV equations possess two expansion families. The first one, the so-called principal family, has resonances $k = -1$, 0, 0, 1, 1, 4, 5, 5, 6, with $\alpha = -2$, $\beta = \gamma = -1$. The usual Painlevé expansion gives a local representation of the general solution.

While for the second expansion branch, the leading-order analysis uniquely gives

$$\begin{aligned} &\alpha = \beta = \gamma = -2, \\ &u_0 = -2f_x^2,\ v_0 = f_x^2, \end{aligned} \tag{5}$$

and w_0 is an arbitrary function. Collecting terms containing u_k, v_k, and w_k, the recursion relations for u_k, v_k, and w_k are found to be

$$\begin{aligned} &-\frac{1}{4}f_x^3(k-4)(k^2-5k-18)u_k + 6f_x^3(k-4)v_k \\ &= F_1(u_{k-1},\ v_{k-1},\ w_{k-1},\ \ldots,\ w_0,\ f_t,\ f_x,\ f_{xx},\ \ldots), \end{aligned} \tag{6}$$

$$\begin{aligned} &-6f_x^3u_k + \frac{1}{2}f_x^3(k^3-9k^2+14k)v_k \\ &= F_2(u_{k-1},\ v_{k-1},\ w_{k-1},\ \ldots,\ w_0,\ f_t,\ f_x,\ f_{xx},\ \ldots), \end{aligned} \tag{7}$$

$$\begin{aligned} &6f_xw_0u_k + \frac{1}{2}f_x^3(k^3-9k^2+14k)w_k \\ &= F_3(u_{k-1},\ v_{k-1},\ w_{k-1},\ \ldots,\ w_0,\ f_t,\ f_x,\ f_{xx},\ \ldots), \end{aligned} \tag{8}$$

for $k = 0,\ 1,\ 2,\ \ldots$.

From (6) – (8), putting to zero the coefficient determinant of u_k, v_k, w_k, we find that the resonances occur at

$$k = -2,\ -1,\ 0,\ 2,\ 3,\ 4,\ 6,\ 7,\ 8. \tag{9}$$

The resonance for $k = -1$, corresponds to the arbitrary singularity manifold ($f = 0$). If the GCKdV system is Painlevé integrable, the resonance conditions for $k = -2$, 0, 2, 3, 4, 6, 7, 8 must be identically satisfied such that the other eight arbitrary functions among the u_k, v_k, w_k can be introduced into the general expansions (4).

The whole Painlevé analysis of PDEs is shown to be invariant [4] under an arbitrary homographic transformation of the singularity manifold f. The best expansion function is $\chi = (f_x/f - f_{xx}/2f_x)^{-1}$. Considering the complexity of the GCKdV equations, we simply perform the invariant Painlevé analysis in the gauge $S = 0$, which will greatly shorten the expressions for the coefficients of the expansion. So the Riccati system [4] satisfied by the function χ automatically becomes

$$\chi_x = 1,\ \chi_t = -C + C_x\chi - \frac{1}{2}C_{xx}\chi^2,$$

with $C = -f_t/f_x$, and the meanings of S and C are explained clearly in [4]. Automatically the cross-derivative condition on the Riccati system is $C_{xxx} = 0$.

Then applying the invariant analysis for the GCKdV equations with help of the software Maple, we find

$$k=0: \; u_0=-2,\; v_0=1. \quad k=1: \; u_1=0,\; v_1=0,\; w_1=-w_{0x}. \quad k=2: \; u_2=\frac{1}{3}C,\; v_2=-\frac{2}{3}C+\frac{1}{2}w_0.$$

$$k=3: \; v_3=-u_3+\frac{1}{3}C_x-\frac{1}{2}w_{0x},\; w_3=-(u_3+\frac{1}{3}C_x)w_0+\frac{1}{6}w_{0t}+\frac{1}{3}w_{0xxx}+\frac{1}{6}w_{0x}C-w_{2x}.$$

$$k=4: \; v_4=-\frac{1}{2}u_4-\frac{1}{12}C_{xx}+\frac{1}{2}u_{3x}+\frac{1}{4}w_{0xx},$$
$$w_4=(u_3+\frac{1}{6}C_x)w_{0x}+(\frac{1}{2}u_{3x}-\frac{1}{2}u_4+\frac{1}{4}C_{xx})w_0-\frac{5}{24}w_{0xxxx}-\frac{1}{6}w_{0xt}-\frac{1}{6}w_{0xx}C+\frac{1}{2}w_{2xx}.$$

$$k=5: \; u_5=(-\frac{22}{63}C_x+\frac{1}{7}w_{0x}-\frac{20}{21}u_3)C-\frac{2}{63}C_t+\frac{1}{7}w_{0t}-u_{4x}-\frac{1}{2}u_{3xx},$$
$$v_5=(\frac{2}{21}C_x+\frac{8}{21}u_3-\frac{1}{42}w_{0x})C-\frac{1}{12}w_{0xxx}+\frac{1}{2}u_{4x}-\frac{2}{63}C_t-\frac{1}{42}w_{0t},$$
$$w_5=[(-\frac{2}{35}w_{0x}+\frac{8}{21}u_3+\frac{44}{315}C_x)C+\frac{1}{2}u_{4x}+\frac{4}{315}C_t-\frac{2}{35}w_{0t}]w_0+(\frac{1}{2}u_4-\frac{1}{6}C_{xx}$$
$$-\frac{1}{2}u_{3x})w_{0x}+(\frac{1}{15}w_{2x}+\frac{1}{20}w_{0xxx})C+\frac{1}{15}w_{2t}-\frac{1}{6}w_{2xxx}+\frac{3}{40}w_{0xxxxx}+\frac{1}{20}w_{0xxt}-\frac{1}{2}u_3w_{0xx}.$$

$$k=6: \; v_6=(\frac{13}{252}C_x-\frac{1}{21}w_{0x}-\frac{1}{14}u_3)C_x+(\frac{1}{84}u_{3x}+\frac{13}{252}C_{xx}-\frac{1}{21}w_{0xx})C$$
$$-\frac{1}{2}u_6-\frac{1}{12}u_{3t}-\frac{1}{4}u_3^2+\frac{1}{12}u_{3xxx}-\frac{1}{21}w_{0xt}+\frac{1}{48}w_{0xxxx}+\frac{5}{252}C_{xt},$$
$$w_6=[(\frac{1}{140}C_x-\frac{1}{14}u_3-\frac{1}{70}w_{0x})w_0+\frac{1}{180}w_{0xxx}-\frac{1}{72}w_{0t}-\frac{44}{315}w_{0x}C-\frac{1}{15}w_{2x}]C_x$$
$$+[(-\frac{1}{70}w_{0xx}+\frac{1}{84}u_{3x}+\frac{1}{140}C_{xx})C-\frac{1}{2}u_6+\frac{1}{12}u_{3xxx}-\frac{1}{12}u_{3t}-\frac{31}{1260}C_{xt}-\frac{1}{70}w_{0xt}-\frac{1}{4}u_3^2]w_0$$
$$+(-\frac{71}{168}w_{0x}C+\frac{1}{6}w_{0xxx}-\frac{1}{24}w_{0t})u_3+(\frac{1}{4}u_{3x}+\frac{1}{24}C_{xx}-\frac{1}{4}u_4+\frac{1}{72}C^2)w_{0xx}$$
$$+(\frac{2}{35}w_{0x}C+\frac{2}{35}w_{0t}+\frac{1}{840}C_t-\frac{1}{2}u_{4x})w_{0x}+(\frac{1}{36}w_{0xt}+\frac{1}{180}w_{0xxxx}-\frac{1}{15}w_{2xx})C-\frac{1}{15}w_{2xt}$$
$$+\frac{1}{24}w_{2xxxx}+\frac{1}{72}w_{0tt}-\frac{7}{360}w_{0xxxxxx}+\frac{1}{180}w_{0xxxt}.$$

$$k=7: \; u_7=(\frac{13}{42}C_{xx}-u_{3x}-u_4)u_3+(\frac{11}{28}u_{3xx}-\frac{1}{14}w_{0xxx}-\frac{1}{12}u_{4x})C+(\frac{13}{21}u_{3x}+\frac{59}{126}C_{xx}-\frac{1}{3}u_4-\frac{1}{7}w_{0xx})C_x$$
$$-u_{6x}+\frac{5}{24}u_{3xxxx}-\frac{1}{14}w_{0xxt}-\frac{1}{12}u_{4t}-\frac{1}{14}w_{0x}C_{xx}+\frac{1}{3}u_{4xxx}-\frac{1}{12}u_{3xt}+\frac{1}{504}C_{xxt},$$
$$v_7=[(-\frac{1}{60}w_{0x}+\frac{1}{45}C_x)C+\frac{1}{45}C_t-\frac{1}{60}w_{0t}]w_0+(-\frac{23}{252}C_{xx}+\frac{2}{3}u_{3x}-\frac{4}{63}C^2+\frac{1}{6}u_4)u_3$$
$$+(-\frac{10}{189}C_x+\frac{2}{63}w_{0x})C^2+(-\frac{1}{24}u_{4x}-\frac{41}{168}u_{3xx}+\frac{1}{30}w_{2x}+\frac{3}{70}w_{0xxx}+\frac{2}{63}w_{0t}-\frac{2}{63}C_t)C$$
$$+(-\frac{67}{252}u_{3x}-\frac{109}{378}C_{xx}+\frac{1}{18}u_4+\frac{5}{42}w_{0xx})C_x+\frac{1}{30}w_{2t}-\frac{1}{240}w_{0xxxxx}+\frac{3}{70}w_{0xxt}-\frac{11}{1008}C_{xxt}$$
$$-\frac{1}{24}u_{4t}+\frac{1}{2}u_{6x}-\frac{1}{6}u_{4xxx}+\frac{5}{84}w_{0x}C_{xx}+\frac{1}{24}u_{3xt}-\frac{1}{8}u_{3xxxx}.$$

The functions w_0, w_2, u_3, u_4, u_6, and w_7, not occurring in the resonances conditions, are arbitrary functions for $j=0,\ldots 7$. For resonance $k=8$, u_8 is another arbitrary function, while v_8 and w_8 are functions of C, w_0, w_2, u_3, u_4, u_6, w_7, u_8, and their different-order derivatives. Because of the complexity, here we won't give out the complete expressions of v_8 and w_8.

Based on the previous results, it is obvious that except f only seven other arbitrary functions are found, which also means that using the singular manifold method fails to find out an arbitrary coefficient for resonance $k=-2$. In other words, the general Painlevé expansion (4) only represents a particular solution.

Observing the characters of the resonances (9), we get that the considered GCKdV system has a maximal but nonprincipal family. In order to solve this problem, in the next section we use the perturbative Painlevé analysis [7] to find an arbitrary function for the resonance $k=-2$, which extends the particular solution (4) into a general one.

3. Perturbative Painlevé Analysis for GCKdV Equations

In [7], R. Conte, A. P. Fordy and A. Picking have further improved the Painlevé test such that negative indices can be treated. In this section, we seek a Laurent expansion for a solution near the solution obtained by the standard Painlevé method in Section 2. We do this by considering a so-called perturbative expansion. For a nonprincipal but maximal Painlevé family, the perturbation extends the particular solution into a representation of the general solution.

Let us denote the Painlevé expansion (4) as ($u^{(0)}$, $v^{(0)}$, $w^{(0)}$), and look for a nearby solution formally represented by an infinite perturbative series in a small parameter ε not occurring in the equations itself

$$u=\sum_{n=0}^{\infty}\varepsilon^n u^{(n)},\ v=\sum_{n=0}^{\infty}\varepsilon^n v^{(n)},\ w=\sum_{n=0}^{\infty}\varepsilon^n w^{(n)}. \quad (10)$$

At zeroth order, the expansion depends on the eight arbitrary functions (f, $w_0^{(0)}$, $w_2^{(0)}$, $u_3^{(0)}$, $u_4^{(0)}$, $u_6^{(0)}$, $w_7^{(0)}$, $u_8^{(0)}$). From the invariant Panilevé analysis, the truncation at the constant level is

$$u^{(0)}=-\frac{2}{f^2}+\frac{1}{3}C+O(f),$$
$$v^{(0)}=\frac{1}{f^2}+\frac{1}{2}w_0^{(0)}-\frac{2}{3}C+O(f),$$
$$w^{(0)}=\frac{w_0^{(0)}}{f^2}-\frac{w_{0x}^{(0)}}{f}+w_2^{(0)}+O(f). \quad (11)$$

At first order, we consider the assumption

$$\begin{aligned}&u=u^{(0)}+\varepsilon u^{(1)},\ v=v^{(0)}+\varepsilon v^{(1)},\\ &w=w^{(0)}+\varepsilon w^{(1)},\\ &u^{(1)}=\frac{U_1}{f^3}+O(\frac{1}{f^2}),\ v^{(1)}=\frac{V_1}{f^3}+O(\frac{1}{f^2}),\\ &w^{(1)}=\frac{W_1}{f^3}+O(\frac{1}{f^2}).\end{aligned} \quad (12)$$

Substituting ansatz (12) into the left hand side of GCKdV equations (3), selecting terms containing ε, and then putting to zero the coefficients of f^{-6}, i. e., the lowest-order terms of the expansion variable f, we obtain

$$15U_1+30V_1=0, \quad (13)$$
$$-6U_1-12V_1=0, \quad (14)$$
$$-12W_1-6U_1w_0^{(0)}=0. \quad (15)$$

Solving the equations (13)–(15), we get

$$V_1=-\frac{1}{2}U_1,\ W_1=-\frac{1}{2}U_1w_0^{(0)}, \quad (16)$$

and U_1 is an arbitrary function, which turns up as the ninth arbitrary function from the resonances for $k=-2$.

Therefore all of the resonance conditions with nine arbitrary functions are identically satisfied, which simultaneously means that the particular solution has already been extended into a representation of the general solution by the perturbative Painlevé approach.

4. Exact Solutions for GCKdV Equations

In this section according to the results of the previous Painlevé studies, we investigate several new types of exact solutions for the GCKdV equations. First, we use the standard truncation of the WTC expansion to find some exact solutions of the GCKdV system.

4.1. Standard Truncation Expansion

The standard truncation form of the WTC Painlevé expansion reads

$$\begin{aligned}&u=\frac{u_0}{f^2}+\frac{u_1}{f}+u_2,\ v=\frac{v_0}{f^2}+\frac{v_1}{f}+v_2,\\ &w=\frac{w_0}{f^2}+\frac{w_1}{f}+w_2,\end{aligned} \quad (17)$$

which evidently is a Bäcklund transformation if $\{u_2, v_2, w_2\}$ is selected as a known solution of the GCKdV equations (3). For simplicity, we fix the seed solution as

$$u_2 = v_2 = w_2 = 0. \tag{18}$$

Substituting (17) with (18) into (3) and putting all the coefficients of the different powers of f to zero, we have

$$u_0 = -2f_x^2,\ v_0 = f_x^2,\ u_1 = 2f_{xx},\ v_1 = -f_{xx},$$
$$w_1 = \frac{f_{xx}w_0 - f_x w_{0x}}{f_x^2},$$

and w_0 is an arbitrary function. A new soliton solution can be obtained by choosing proper expressions of the expansion variable f and the arbitrary function w_0.

For $f = 1 + \mathrm{e}^{(kx+\omega t)}$ and $w_0 = af_{xx}$, the solution is $u = \frac{1}{2}k^2\mathrm{sech}^2[(kx+\omega t)/2]$, $v = -\frac{1}{4}k^2\mathrm{sech}^2[(kx+\omega t)/2]$, $w = \frac{a}{4}k^2\mathrm{sech}^2[(kx+\omega t)/2]$. The substitution of the above solution in (3) gives that $a = -\frac{1}{2}k^2$, and $\omega = -\frac{1}{2}k^3$.

4.2. Periodic Jacobi Elliptic Function Expansion

Recently Liu et al. [10] used three Jacobi elliptic functions (sn(ξ), cn(ξ), and dn(ξ)) to construct exact periodic solutions of some nonlinear evolution equations. The expansion method has been further developed into extended ones and also widely applied by Yan [11], Fan [12], Ye [13], etc. Two types of periodic wave solutions with Jacobi elliptic function sn(ξ) for the considered GCKdV system has been given by Fan [14] in the form

$$u(\xi) = \sum_{i=0}^{p} a_i \mathrm{sn}^i(\xi),\ v(\xi) = \sum_{i=0}^{q} b_i \mathrm{sn}^i(\xi),$$
$$w(\xi) = \sum_{i=0}^{m} c_i \mathrm{sn}^i(\xi), \tag{19}$$

where $\xi = \alpha x - \omega t$, a_i, b_i, c_i, α, and ω are constants to be determined later. The parameters p, q, and m can be determined by the homogeneous balancing method.

In a similar way, we can obtain periodic wave solutions for the GCKdV equations by taking the expansion functions as cn(ξ), dn(ξ), and $\frac{\mathrm{sn}(\xi)}{\mathrm{dn}(\xi)}$.

(i) For the Jacobi elliptic function cn(ξ), we obtain

$$u_1 = \frac{\omega + 2\alpha^3(1-2k^2)}{3\alpha} + 2\alpha^2k^2\mathrm{cn}^2(\xi),$$
$$v_1 = b_0 \pm \alpha^2k^2\mathrm{cn}^2(\xi), \tag{20}$$
$$w_1 = c_0 + \frac{2}{3}\alpha k^2[\alpha^3(2k^2-1) - 2\omega \pm 3b_0\alpha]\mathrm{cn}^2(\xi),$$

with α, ω, b_0, c_0 being arbitrary constants. Here we use k to denote the elliptic modulus parameter $(0 \le k \le 1)$, and

$$u_2 = \frac{\alpha^4k^2(1-2k^2) + 2b_1^2}{4\alpha^2k^2} + \alpha^2k^2\mathrm{cn}^2(\xi),$$
$$v_2 = b_0 + b_1\mathrm{cn}(\xi),\ w_2 = c_0 + 2b_0b_1\mathrm{cn}(\xi), \tag{21}$$

where $\xi = \alpha x - \frac{\alpha^4k^2(1-2k^2)+6b_1^2}{4\alpha k^2}t$, α, c_0, $b_0 \neq 0$ and $b_1 \neq 0$ are arbitrary constants.

(ii) Taking dn(ξ) as the expansion variable, we find

$$u_1 = \frac{\omega + 2\alpha^3(k^2-2)}{3\alpha} + 2\alpha^2\mathrm{dn}^2(\xi),$$
$$v_1 = b_0 \pm \alpha^2\mathrm{dn}^2(\xi), \tag{22}$$
$$w_1 = c_0 + \frac{2}{3}\alpha[\alpha^3(2-k^2) \pm 3b_0\alpha - 2\omega]\mathrm{dn}^2(\xi),$$

with α, ω, b_0, c_0 being arbitrary constants, and

$$u_2 = \frac{\alpha^4(k^2-2) + 2b_1^2}{4\alpha^2} + \alpha^2\mathrm{dn}^2(\xi),$$
$$v_2 = b_0 + b_1\mathrm{dn}(\xi),\ w_2 = c_0 + 2b_0b_1\mathrm{dn}(\xi), \tag{23}$$

where $\xi = \alpha x - \frac{\alpha^4(k^2-2)+6b_1^2}{4\alpha}t$, and all these parameters possess the same definition as in the solution (21).

(iii) And for $\frac{\mathrm{sn}(\xi)}{\mathrm{dn}(\xi)}$, we have the solutions

$$u_1 = a_0 + 2\alpha^2k^2(1-k^2)\frac{\mathrm{sn}^2(\xi)}{\mathrm{dn}^2(\xi)},$$
$$v_1 = b_0 \mp \alpha^2k^2(1-k^2)\frac{\mathrm{sn}^2(\xi)}{\mathrm{dn}^2(\xi)}, \tag{24}$$
$$w_1 = c_0 - 2\alpha^2k^2(1-k^2) \cdot (2\alpha^2k^2 - \alpha^2 + 2a_0 \pm b_0)\frac{\mathrm{sn}^2(\xi)}{\mathrm{dn}^2(\xi)},$$

with $\xi = \alpha x - \alpha[3a_0 + 2\alpha^2(2k^2-2)]t$, a_0, b_0, c_0 and α being arbitrary constants, and

$$u_2 = \frac{\alpha^4k^2(1+2k^4-3k^2)+2b_1^2}{\alpha^2k^2(1-k^2)} + \alpha^2k^2(1-k^2)\frac{\mathrm{sn}^2(\xi)}{\mathrm{dn}^2(\xi)},$$
$$v_2 = b_0 + b_1\frac{\mathrm{sn}(\xi)}{\mathrm{dn}(\xi)},\ w_2 = c_0 + 2b_0b_1\frac{\mathrm{sn}(\xi)}{\mathrm{dn}(\xi)}, \tag{25}$$

where $\xi = \alpha x - \frac{\alpha^4\ k^2(1+2k^4-3k^2)+6b_1^2}{\alpha k^2(1-k^2)} t,\ \alpha,\ c_0,\ b_0 \neq 0$ and $b_1 \neq 0$ are arbitrary constants.

Since $\mathrm{dn}(\xi) \rightarrow \mathrm{sech}(\xi)$, and $\mathrm{sn}(\xi)$, $\mathrm{cn}(\xi) \rightarrow \tanh(\xi)$ as $k \rightarrow 1$, the periodic solutions (20)–(25) exactly converge to soliton solutions.

(iv) Actually, the summation exponent i of the expansion (19) can range from $-p \rightarrow p$, $-q \rightarrow q$, and $-m \rightarrow m$, i. e. the expansion can be rewritten as

$$u(\xi) = \sum_{i=-p}^{p} a_i\phi^i,\ v(\xi) = \sum_{i=-q}^{q} b_i\phi^i,\ w(\xi) = \sum_{i=-m}^{m} c_i\phi^i. \tag{26}$$

The following is a simple proof of the expansion (26) by randomly choosing ϕ as $\mathrm{dn}(\xi)$. The parameters p, q, and m can be determined by the same method as in the previous section. Thus the system admits the following ansatz:

$$\begin{aligned} u &= a_0 + a_1\mathrm{dn}(\xi) + a_2\mathrm{dn}^2(\xi) + \frac{a_3}{\mathrm{dn}^2(\xi)} + \frac{a_4}{\mathrm{dn}(\xi)}, \\ v &= b_0 + b_1\mathrm{dn}(\xi) + b_2\mathrm{dn}^2(\xi) + \frac{b_3}{\mathrm{dn}^2(\xi)} + \frac{b_4}{\mathrm{dn}(\xi)}, \\ w &= c_0 + c_1\mathrm{dn}(\xi) + c_2\mathrm{dn}^2(\xi) + \frac{c_3}{\mathrm{dn}^2(\xi)} + \frac{c_4}{\mathrm{dn}(\xi)}. \end{aligned} \tag{27}$$

Substituting (27) into (3) and using the software Maple, we obtain three periodic wave solutions

$$\begin{aligned} u_1 &= a_0 + \frac{2\alpha^2(1-k^2)}{\mathrm{dn}^2(\xi)},\ v_1 = b_0 \pm \frac{\alpha^2(1-k^2)}{\mathrm{dn}^2(\xi)}, \\ w_1 &= c_0 + \frac{2\alpha^2(1-k^2)[\alpha^2(k^2-2) \pm b_0 - 2a_0]}{\mathrm{dn}^2(\xi)}; \end{aligned} \tag{28}$$

$$\begin{aligned} u_2 &= \frac{\alpha^4(3k^2-2-k^4)+2b_4^2}{4\alpha^2(1-k^2)} + \frac{\alpha^2(1-k^2)}{\mathrm{dn}^2(\xi)}, \\ v_2 &= b_0 + \frac{b_4}{\mathrm{dn}(\xi)},\ w_2 = c_0 + \frac{2b_0b_4}{\mathrm{dn}(\xi)}; \end{aligned} \tag{29}$$

$$\begin{aligned} u_3 &= \frac{2\alpha^2(1-k^2)}{\mathrm{dn}^2(\xi)} + a_0 + 2\alpha^2\mathrm{dn}^2(\xi), \\ v_3 &= \frac{\pm\alpha^2(1-k^2)}{\mathrm{dn}^2(\xi)} + b_0 \pm \alpha^2\mathrm{dn}^2(\xi), \\ w_3 &= \frac{2\alpha^2(1-k^2)[\alpha^2(k^2-2) \pm b_0 - 2a_0]}{\mathrm{dn}^2(\xi)} \\ &\quad + c_0 + 2\alpha^2[\alpha^2(k^2-2) - 2a_0 \pm b_0]\mathrm{dn}^2(\xi), \end{aligned} \tag{30}$$

where a_0, c_0, $b_0 \neq 0$, and $b_4 \neq 0$ are arbitrary constants.

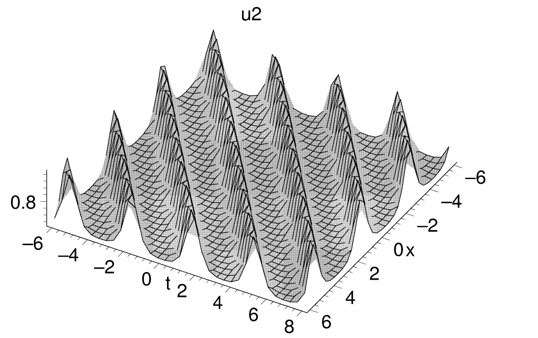

Fig. 1. Plot of the periodic Jacobi elliptic solution u_2 (23), with $\alpha = 1,\ b_1 = 1,\ k = 0.8$.

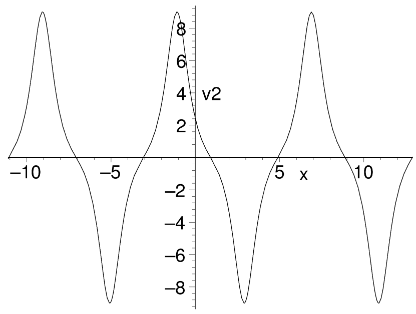

Fig. 2. Plot of the periodic solution v_2 (25) at $t = 2$, for $m = 0.8$, $\alpha = 1$, $b_0 = 0$, and $b_1 = 1$.

Taking $\mathrm{dn}(\xi)$ as the expansion variable, the solution u_2 (23) is plotted in Fig. 1 by choosing specific convenient values of the arbitrary constants α, b_1 and the modulus parameter m. It is evidently shown in Fig. 1 that u_2 is really a periodic solution involving time t. And the solution v_2 (25) is plotted in Figure 2. From the picture we can find that v_2 (25) is a periodic function.

4.3. Extended Truncation Expansion

Because the singularity manifold in the usual Painlevé analysis is arbitrary, one may expand a field in many different forms. Starting from some different expansions, one may take different truncation procedures to get additional solutions. In [6], S. Y. Lou has introduced a simple new expansion for many known integrable and nonintegrable models. We look for the solutions of the form

$$\begin{aligned} u &= \frac{a_0}{f^2} + \frac{a_1}{f} + a_2 + a_3 f + a_4 f^2, \\ v &= \frac{b_0}{f^2} + \frac{b_1}{f} + b_2 + b_3 f + b_4 f^2, \end{aligned} \tag{31}$$

$$w=\frac{h_0}{f^2}+\frac{h_1}{f}+h_2+h_3 f+h_4 f^2,$$

where a_i, b_i, h_i are some arbitrary undetermined constants, and the expansion variable $f(x,t)$ satisfies the Riccati system

$$f_x=c_0+c_1 f+c_2 f^2,\ f_t=g_0+g_1 f+g_2 f^2. \quad (32)$$

The cross-derivative condition on (32) is

$$g_0=\frac{g_1 c_0}{c_1},\ g_1=\frac{g_2 c_1}{c_2}. \quad (33)$$

Based on the substitution of (33) into (32), the expression of function f can be easily obtained

$$f=\frac{\sqrt{4c_0c_2-c_1^2}\tanh(\frac{\xi}{2}\sqrt{4c_0c_2-c_1^2})-c_1}{2c_2}, \quad (34)$$

with $\xi=x+\frac{g_2}{c_2}t$, and $c_2\neq 0,\ g_2\neq 0,\ c_1\neq 0,\ c_0$ being arbitrary constants.

Substituting the solution ansatz (31) together with the Riccati system (32) and its relevant condition (33) into the GCKdV equations (3), we get four new sets of solutions:

$$u_1=\frac{24b_3^2(8g_2c_2+c_1^2c_2^2-6b_3^2)-16g_2c_2^2(4g_2+c_1^2c_2)-c_1^4c_2^4}{64c_2^6f^2}+\frac{c_1(8c_2g_2+c_1^2c_2^2-12b_3^2)}{8c_2^3f}+\frac{c_2g_2-2b_3^2}{c_2^2}-c_1c_2f-c_2^2f^2,$$

$$v_1=\frac{b_3(12b_3^2-8g_2c_2-c_1^2c_2^2)}{8c_2^4f}+b_3f, \quad (35)$$

$$w_1=\frac{16c_1[b_3^2(16g_2c_2+2c_1^2c_2^2-15b_3^2)-g_2c_2^2(4g_2+c_1^2c_2)]}{64c_2^5f}+h_2+\frac{c_1(8g_2c_2+c_1^2c_2^2-20b_3^2)}{8c_2}f,$$

with $b_3\neq 0$ and h_2 being arbitrary constants.

$$u_2=-\frac{c_0^2}{f^2}-\frac{c_0c_1}{f}-\frac{c_2^2(8c_2c_0+c_1^2)+4b_3^2}{8c_2^2}-c_1c_2f-c_2^2f^2,\quad v_2=\frac{b_3c_0}{c_2f}+b_2+b_3f,$$
$$w_2=\frac{c_0[2b_2b_3c_2-c_1(c_2^3c_0+b_3^2)]}{c_2^2f}+h_2+\frac{[2b_2b_3c_2-c_1(c_2^3c_0+b_3^2)]}{c_2}f, \quad (36)$$

with $b_3\neq 0$, $c_0\neq 0$, b_2 and h_2 being arbitrary constants.

$$u_3=-\frac{c_1^4}{2c_2^2f^2}-\frac{c_1^3}{c_2f}-\frac{9}{8}c_1^2-c_1c_2f-c_2f^2,\quad v_3=\pm\frac{c_1^4}{4c_2^2f^2}\pm\frac{c_1^3}{2c_2f}+b_2\pm c_1c_2f,$$
$$w_3=\frac{c_1^4(\pm b_2-c_1^2)}{2c_2^2f^2}+\frac{c_1^3(\pm b_2-c_1^2)}{c_2f}+h_2+2c_1c_2(\pm b_2-c_1^2)f, \quad (37)$$

where b_2 and h_2 are arbitrary constants.

$$u_4=-\frac{c_1^4}{4c_2^2f^2}-\frac{c_1^3}{2c_2f}-\frac{9}{8}c_1^2-2c_1c_2f-2c_2^2f^2,\quad v_4=\pm\frac{c_1^3}{2c_2f}+b_2\pm c_1c_2f\pm c_2^2f^2,$$
$$w_4=\frac{c_1^3(\pm b_2-c_1^2)}{c_2f}+h_2+2c_1c_2(\pm b_2-c_1^2)f+2c_2^2(\pm b_2-c_1^2)f^2, \quad (38)$$

where b_2 and h_2 are arbitrary constants. Simultaneously, all the other parameters a_i, b_i, and h_i, not arising in the solution forms (35) – (38), are taken as zero.

5. Summary

In this paper, the standard and the perturbative Painlevé analysis are used to study the generalized coupled Korteweg-de Vries (GCKdV) equations. The expansion branch at the resonance $k = -2$ is discussed, which obviously shows that the GCKdV system is Painlevé integrable and that one obtains a new Bäcklund transformation different from that of the branch (-1). Using this new Bäcklund transformation, we derive a new soliton solution of the system. By the use of the Jacobi elliptic functions expansion method, several types of periodic wave solutions are obtained. And in the limiting cases for the modulus parameter $k \to 1$, they exactly converge to soliton solutions. Furthermore, applying the method of S. Y. Lou, four types of new solutions of the GCKdV system are also given. Actually the perturbative Painlevé approach can be applied to many other nonlinear wave equations, as long as the nonlinear equations possess nonprinciple but maximal Painlevé expansion families.

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